Coloring triangle-free rectangular frame intersection graphs with O(log log n) colors

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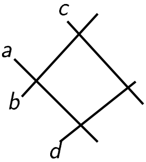
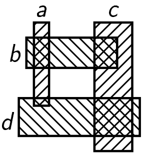
Geometric intersection graphs

$\mathcal{S}$ — a set of compact, arcwise connected objects in the plane.

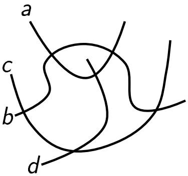
Geometric intersection graph of $\mathcal{S}$:
vertices $\leftrightarrow$ objects,   edges $\leftrightarrow$ intersecting objects.



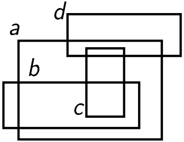
line segments
segment graph



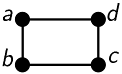
rectangles



curves
string graph



frames



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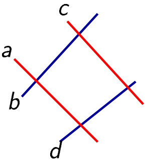

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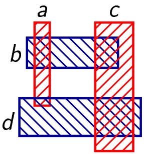

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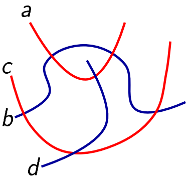

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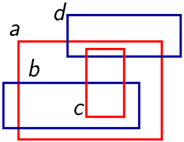

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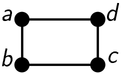


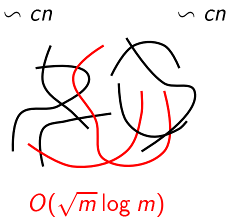
chromatic number - upper bounds

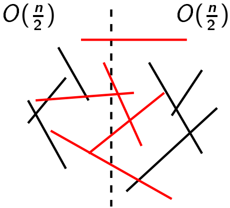
Upper bounds:

- string graphs – $O(\log^{\log \omega} n)$ – Fox, Pach, 2013
  - Separator Theorem for string graphs,
- segment graphs – $O(\log n)$ - Suk, 2012
  - segments piercing a common line have bounded chromatic number,
- rectangles – $O(\omega^2)$ – Asplund and Grünbaum.

Comments:

- nothing better than $O(\log n)$ was known.





chromatic number - lower bounds

Lower bounds:

- $\Omega(\log \log n)$ – Pawlik, Kozik, T.K., Lasoń, Micek, Walczak, Trotter, 2012
  - triangle-free segment/string graphs,
  - objects obtained by horizontal scaling, vertical scaling, and translation of some fixed object (but not rectangles),
- rectangles – 3ω – Kostochka.

Comments:

- obtained via on-line coloring games on intervals in the line!.

result

Theorem (T.K., Pawlik, Walczak, 2012)

Every triangle-free intersection graph of frames with n vertices can be colored with $O(\log \log n)$ colors.

Comments:

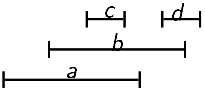
- the first algorithm that beats $O(\log n)$ bound,
- describes precisely the structure of frame intersection graphs,
- uses on-line coloring algorithms.

Problems:

- replace triangle-free with K_d-free,
- extend the method on segment graphs, L-shaped graphs,
- limitations of our method?

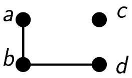
overlap graphs

Overlap (circle) graph:
vertices $\leftrightarrow$ intervals in the line, edges $\leftrightarrow$ overlapping intervals.



Theorem (Kostochka, Kratochvíl, 1997)

Every K_ω-free overlap graph can be colored with $50 \cdot 2^\omega$ colors!



overlap coloring game

K_ω-free overlap coloring game:

- played by Presenter and Algorithm in rounds,
- Presenter builds a K_ω-free overlap graph:
  - one interval per round,
  - presentation order – consistent with left-endpoints relation.

- Algorithm colors intervals (immediately and irrevocably) so that no two overlapping intervals have the same color.

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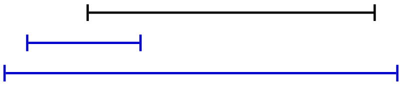

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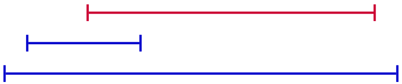

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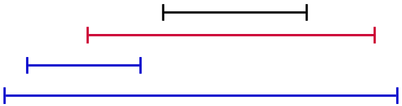

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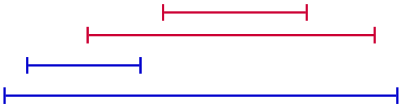

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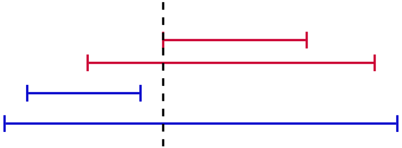

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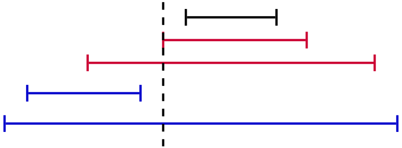

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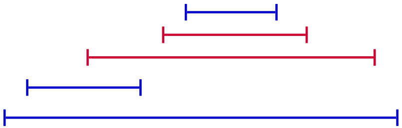

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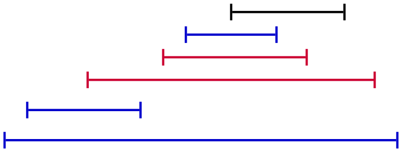

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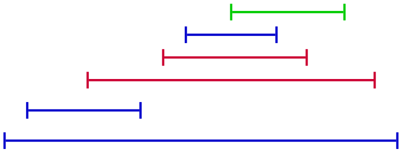

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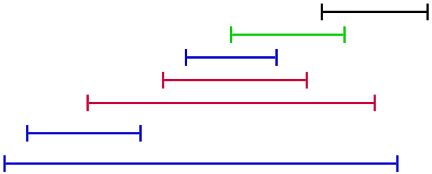

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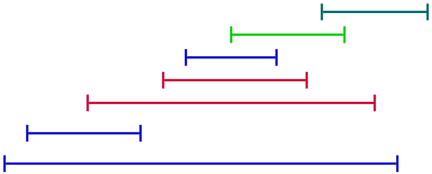

overlap coloring game

Observation (Pawlik, Kozik, T.K., Lasoń, Micek, Walczak, Trotter, 2012)

If Presenter has a strategy to force Algorithm to use c colors in h rounds of K_ω-free overlap coloring game

$$\Downarrow$$

There is a K_ω-free frame intersection graph with $2^{poly(h)}$ vertices and chromatic number at least c.

universal graph

h-universal graph – 'encodes' all possible moves of Presenter in the first h rounds of K_ω-free o.c. game.

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round 1

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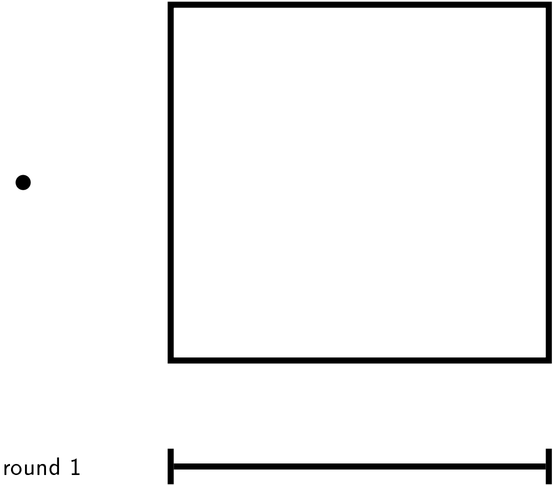

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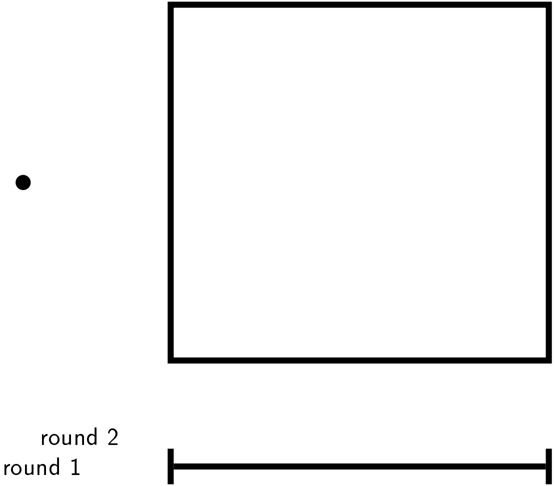

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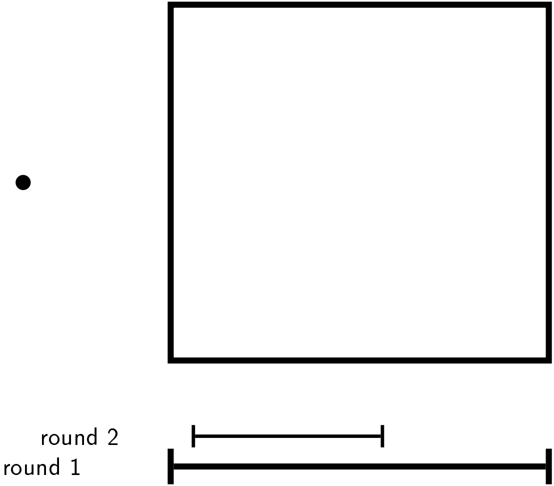

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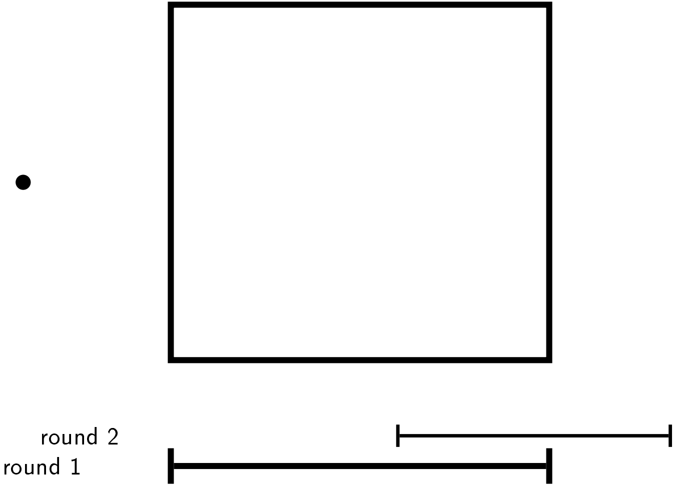

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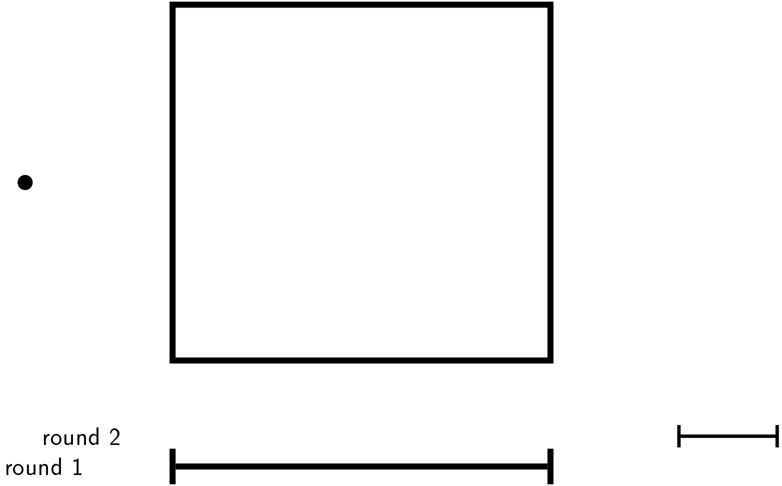

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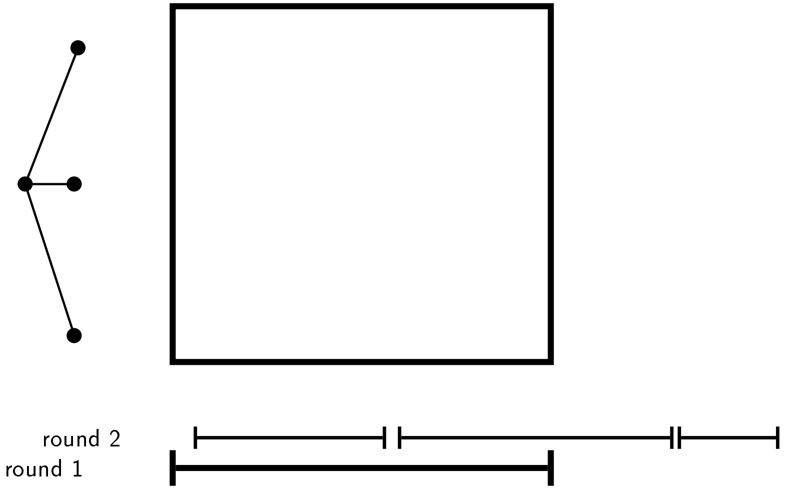

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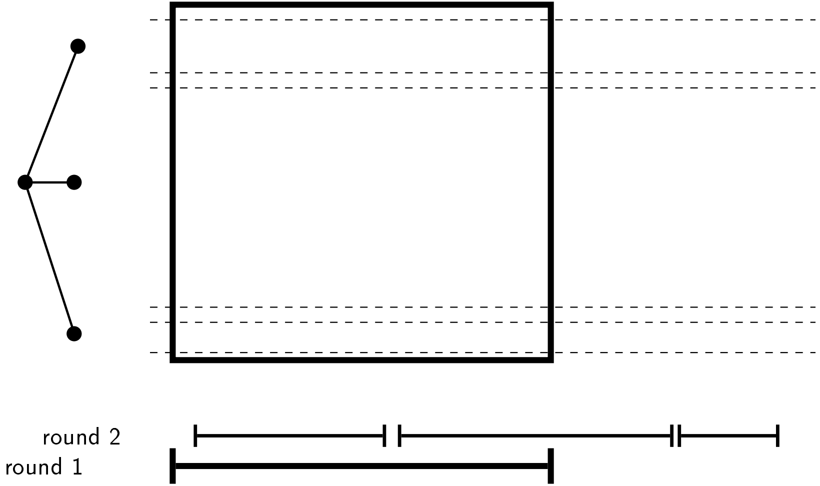

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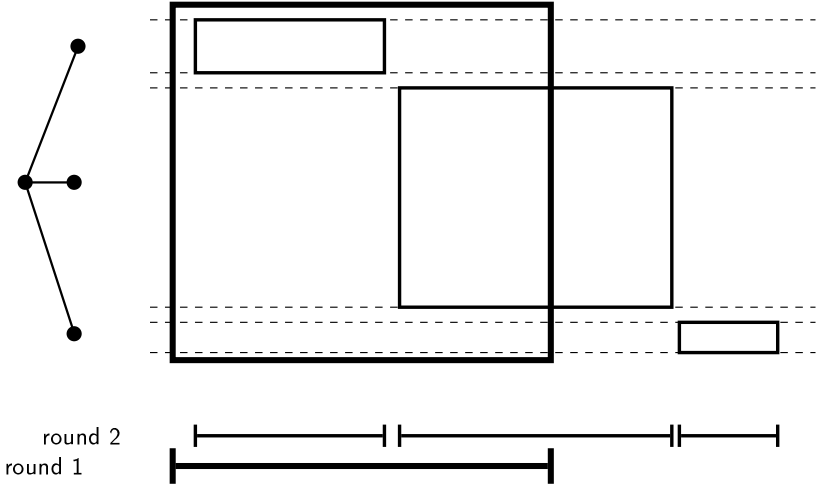

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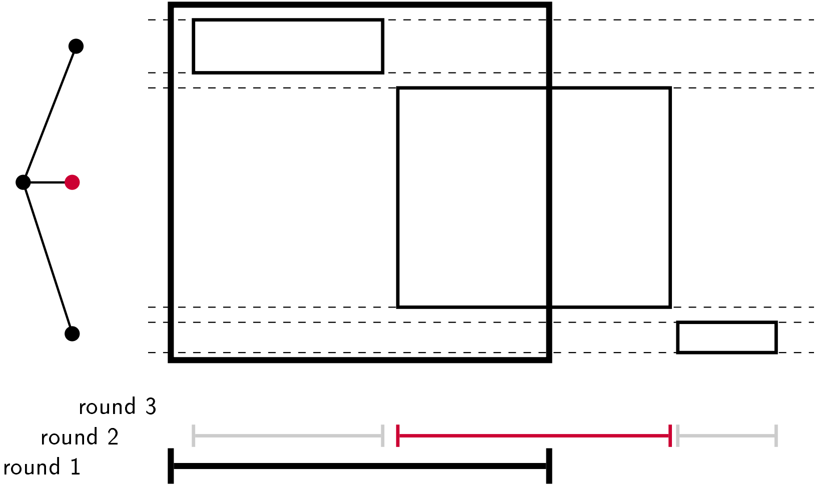

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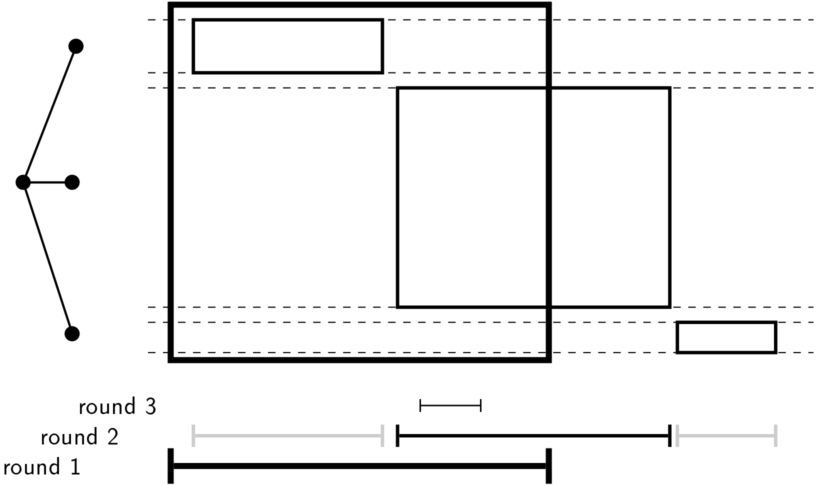

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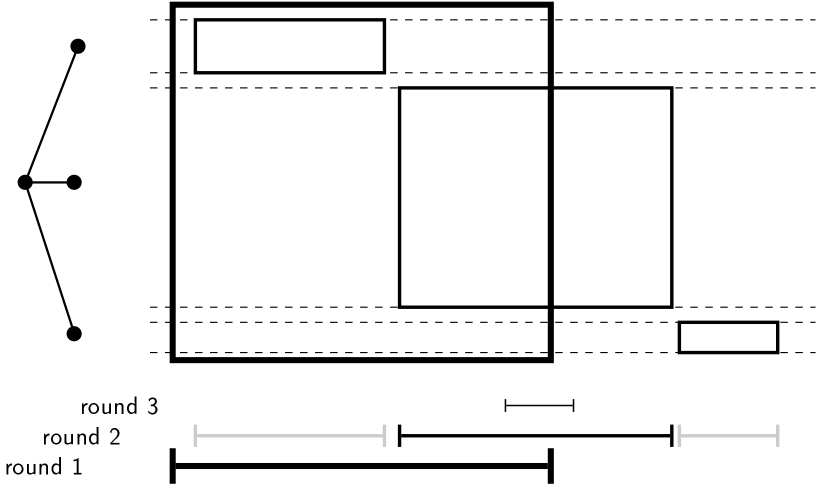

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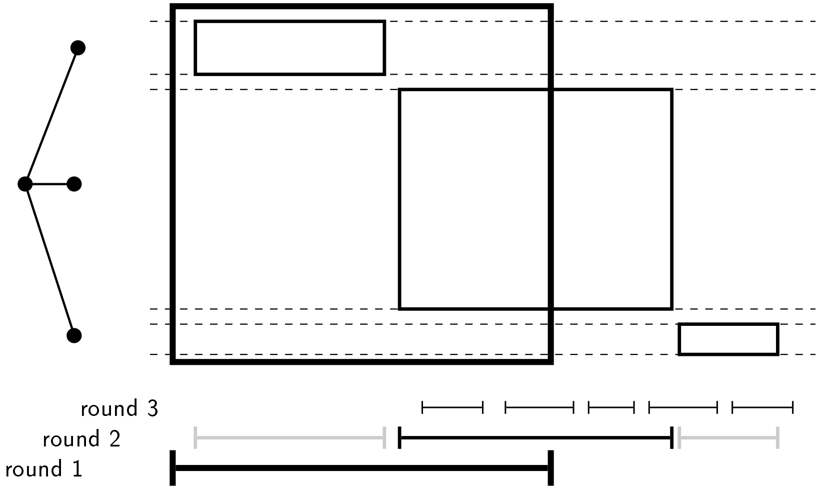

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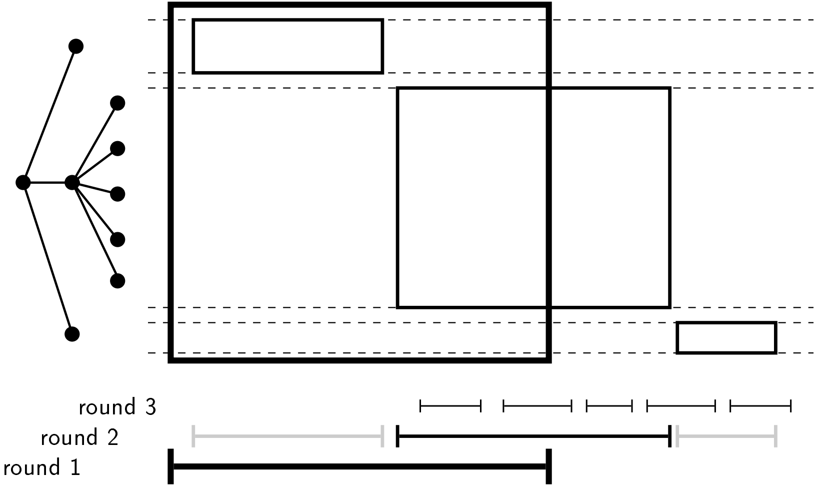

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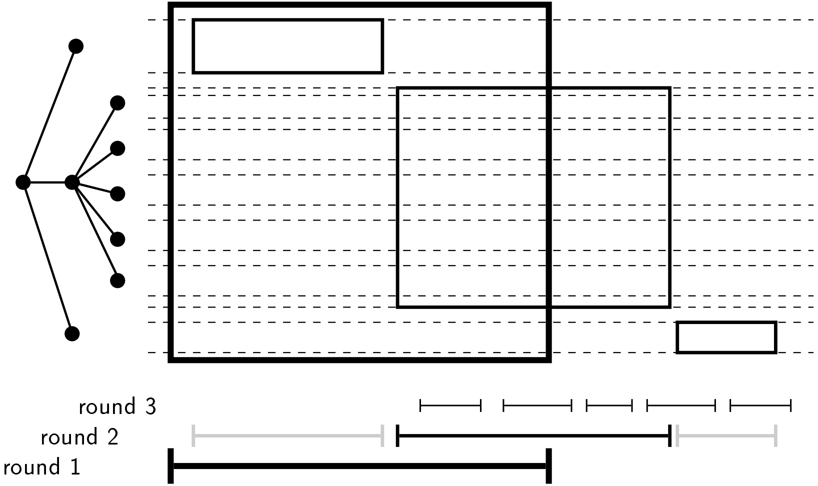

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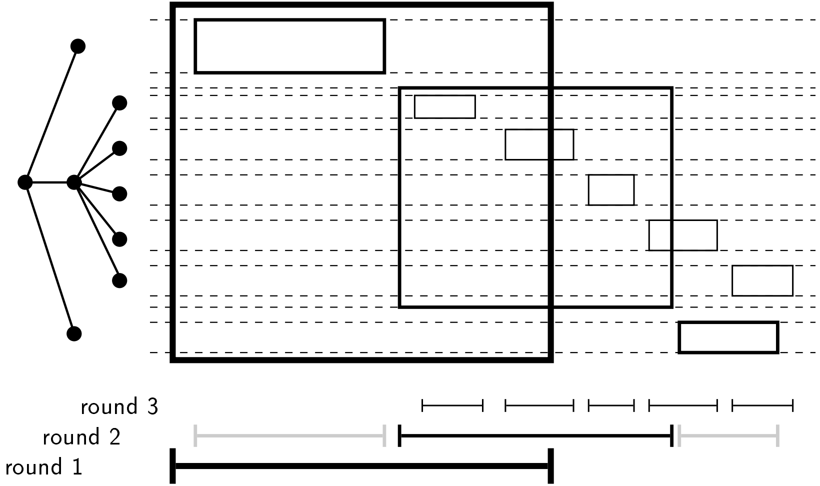

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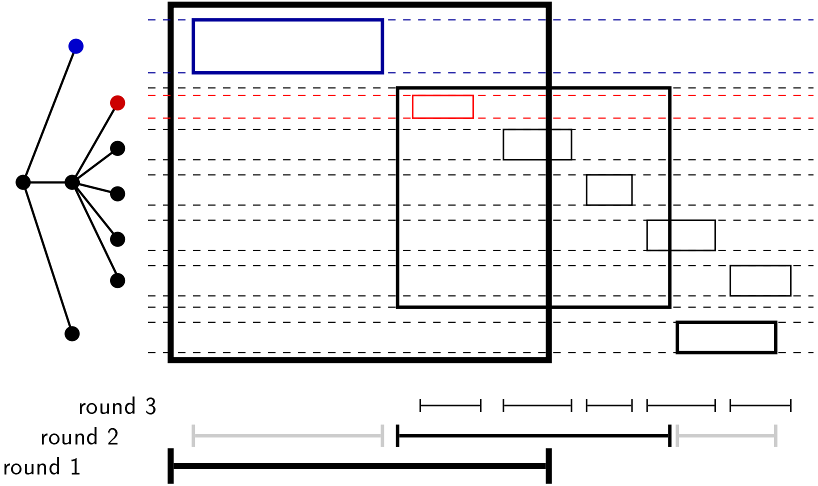

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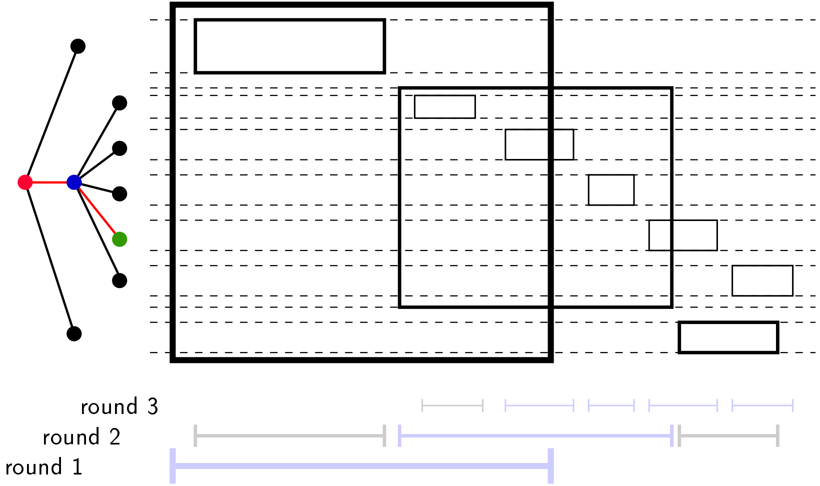

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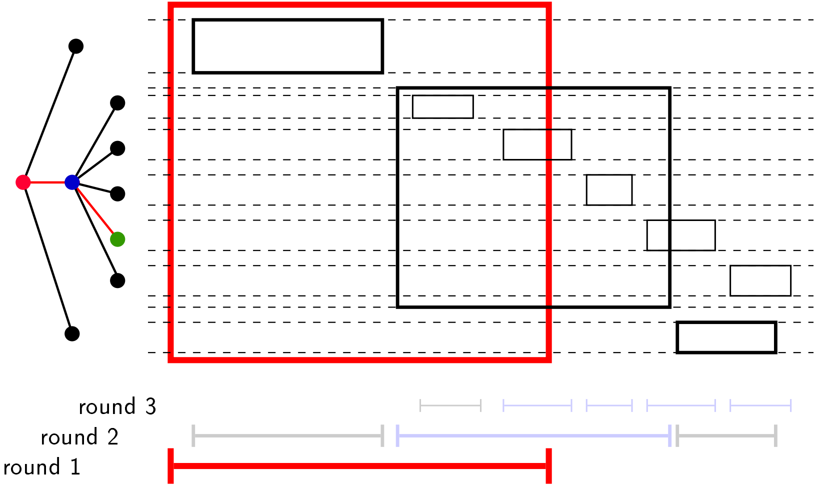

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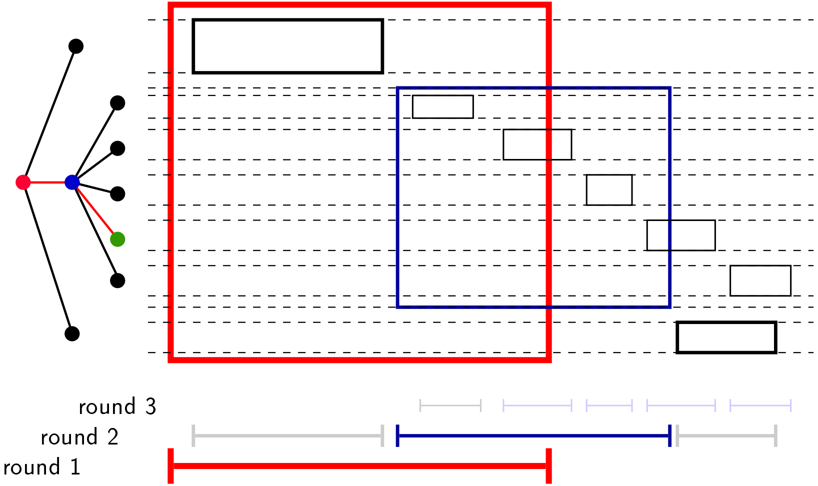

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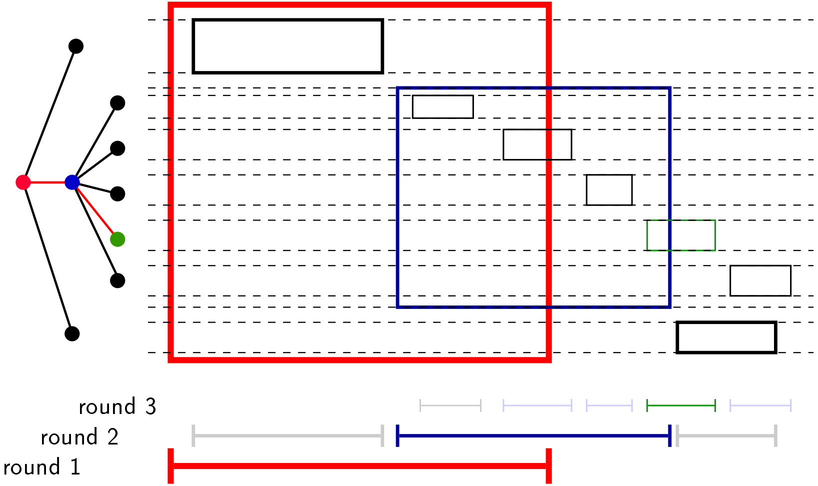

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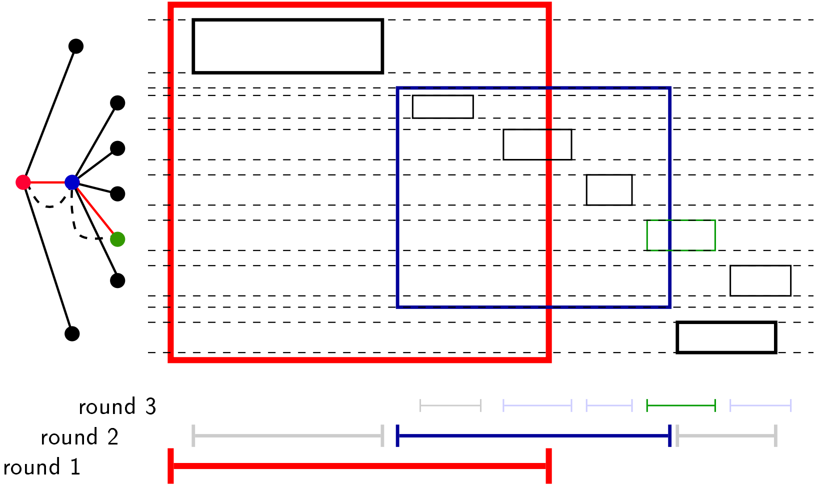

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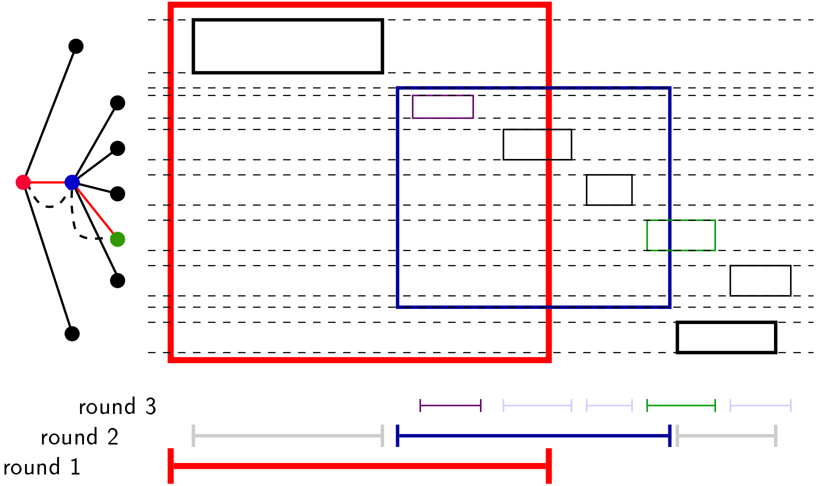

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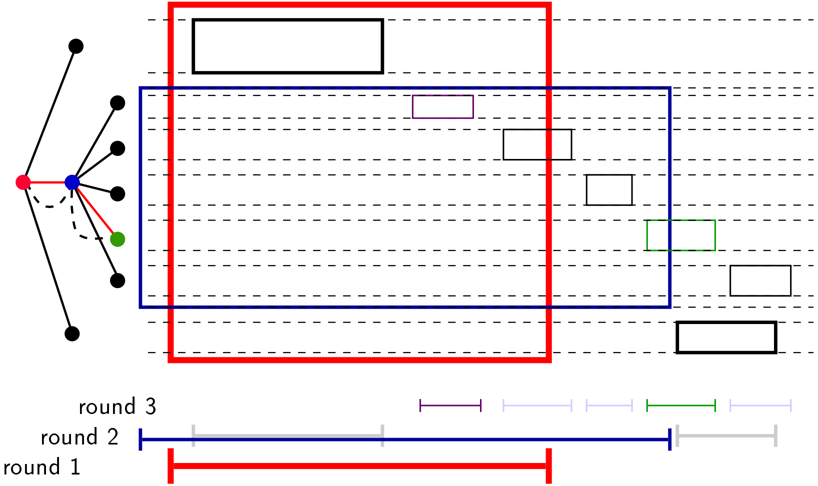
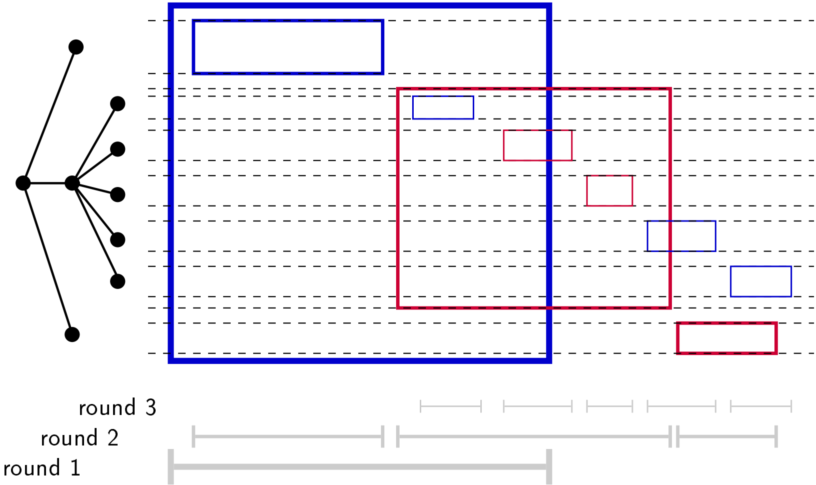


observation - proof

Correspondence: On-line algorithms – Proper colorings.



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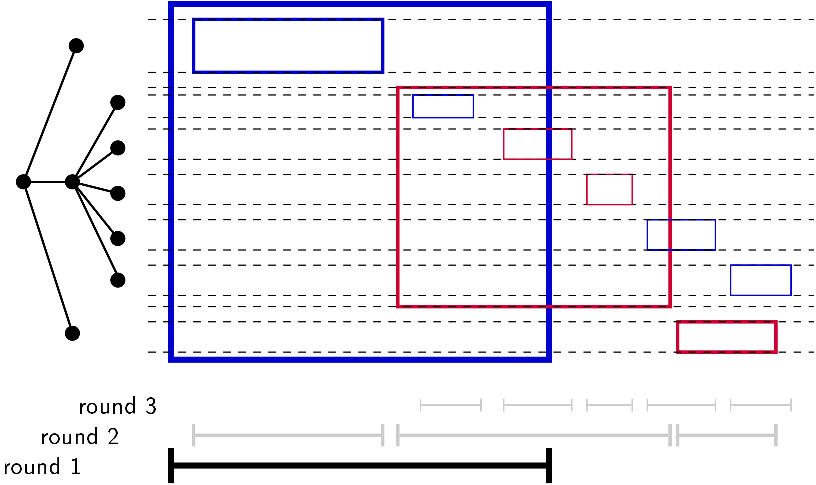

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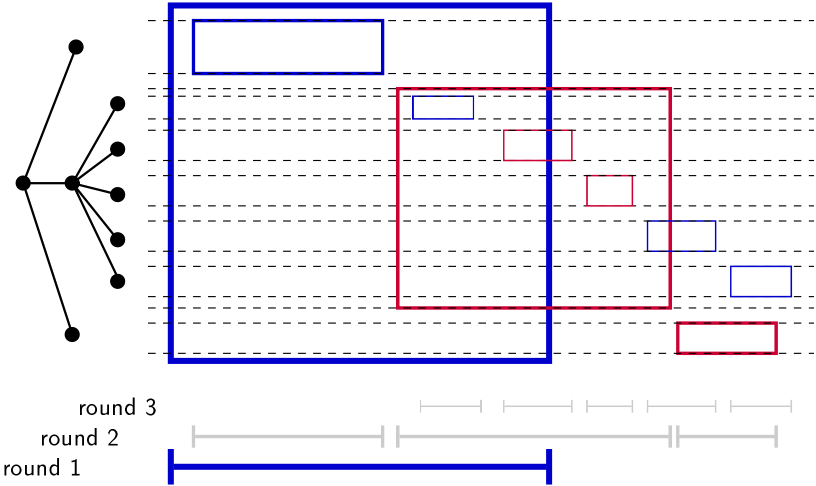

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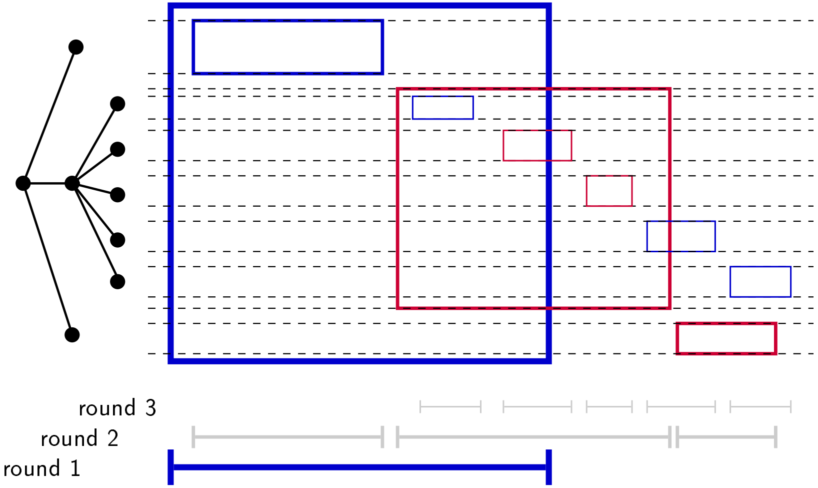

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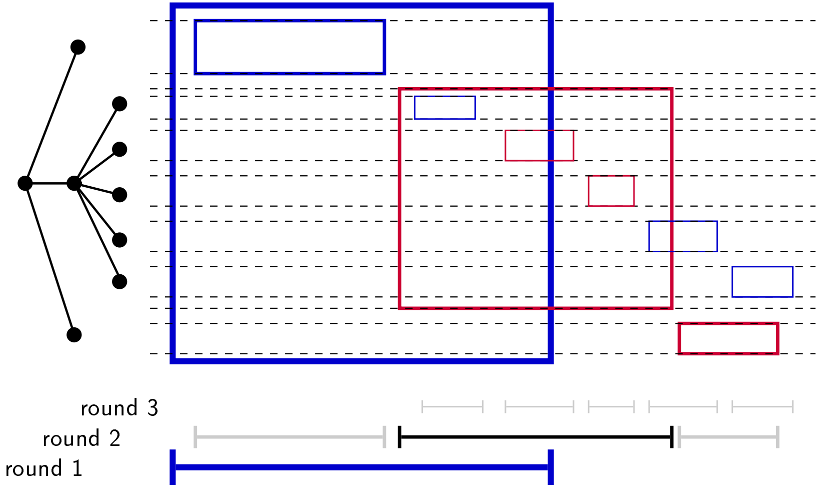

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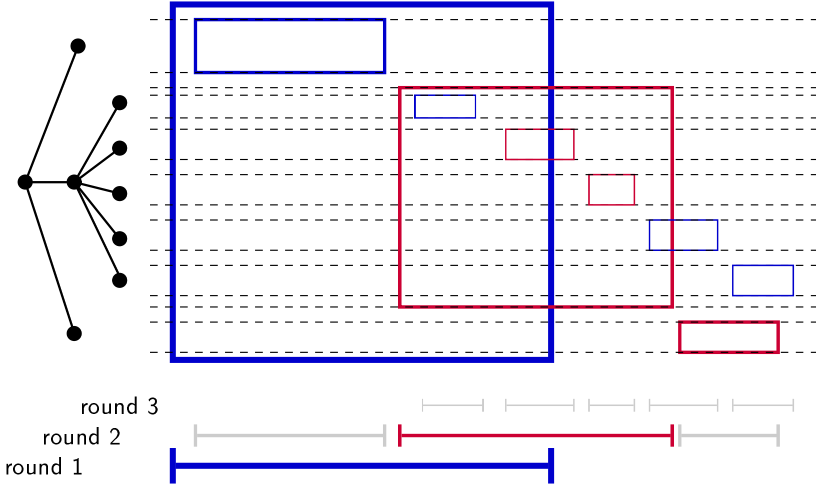

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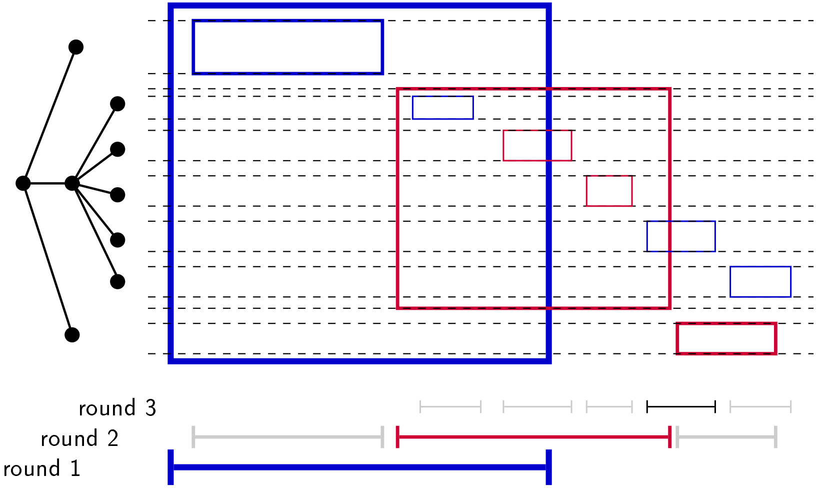

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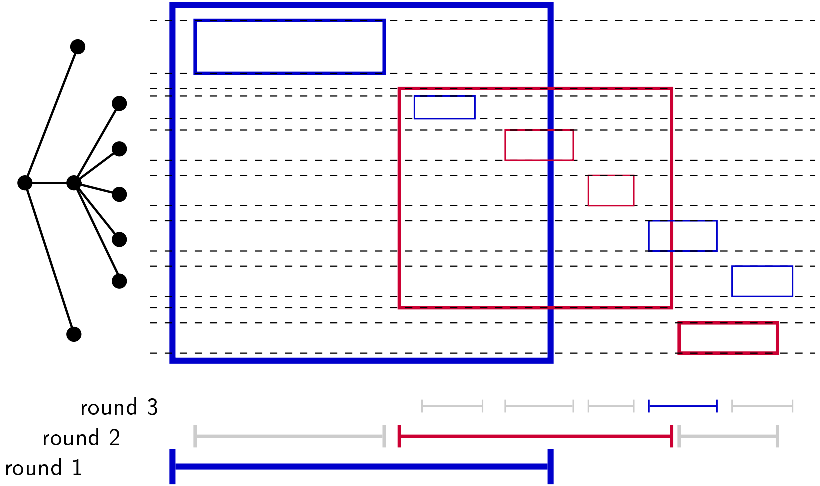


overlap coloring game - lower bounds

Observation (Pawlik, Kozik, T.K., Lasoń,Micek, Walczak, Trotter, 2012)

There is a strategy for Presenter that forces Algorithm to use $\log h$ *colors in* h *rounds of the triangle-free o.c. game.*

Comments:

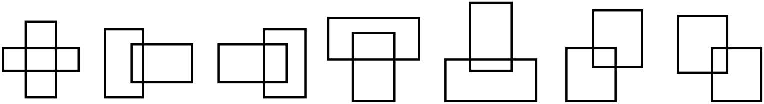
- there are triangle-free frame intersection graphs (universal graphs) with n vertices and chromatic number $\Omega(\log \log n)$,
- universal graphs can be represented also by segments in the plane.

upper bound for frames

On-line algorithms from overlap coloring games are also useful in frame coloring.

frames - intersection types

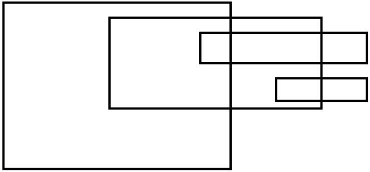
Intersection types:



a crossing rightward- leftward- downward- upward- diagonal intersections
directed

directed families of frames

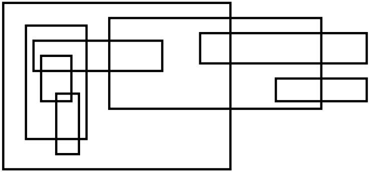
A family $\mathcal{F}$ of frames is rightward-directed (leftward-, upward-, downward-directed) if the intersection of every two frames from $\mathcal{F}$ is rightward-directed (leftward-, upward-, downward-directed).



decomposition theorem

Theorem (T.K., Pawlik, Walczak, 2012)

Every K_ω-free family of frames $\mathcal{F}$ can be partitioned into $f(\omega)$ componentwise directed subfamilies.



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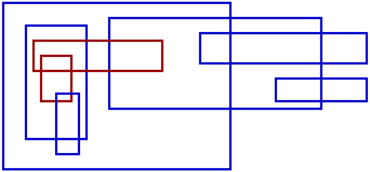
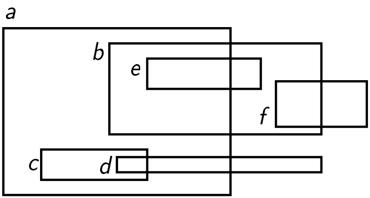


Overlap game graph – a connected, directed component.

- Projections of every two frames on OY-axis are either nested or disjoint,
- encode a 'partial' overlap coloring game (Presenter's power is limited).

Comments:

- encode a 'partial' overlap coloring game (Presenter's power is limited).



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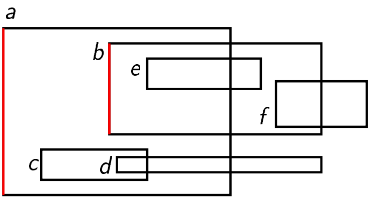

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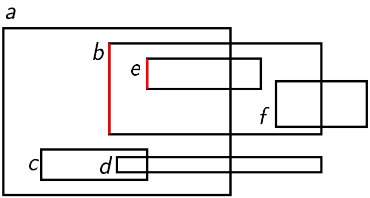

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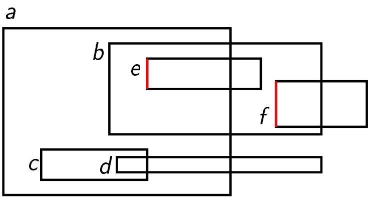

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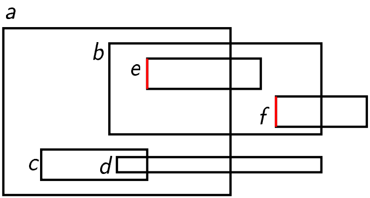

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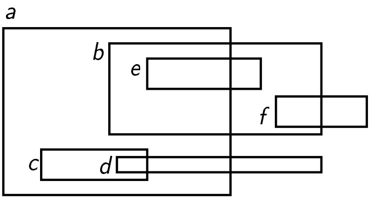

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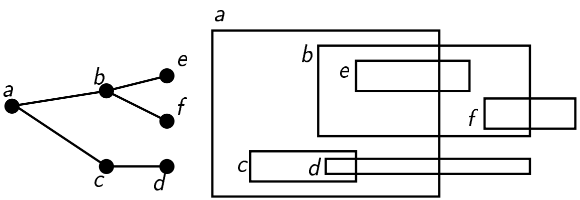
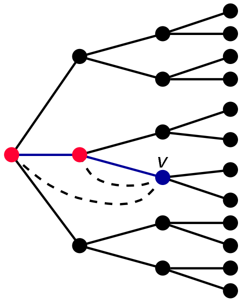


use of on-line algorithms

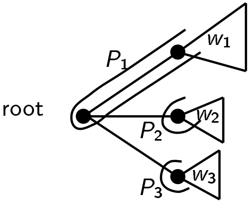
Coloring universal graphs:

- h-universal graph - overlap game graph that encodes the first h rounds of the game, ($2^{\mathrm{poly}(h)}$ vertices)
- use on-line algorithm! $O(\log h)$ colors.
- the chromatic number of universal graph is $O(\log \log n)$.



heavy-light decomposition

- introduced by Sleator and Tarjan, 1983,
- a partition of a rooted tree into heavy paths,
- $w_1 \in P_1 \implies |w_i| \leqslant |w_1|$
- if a path from the root to a leaf intersects k heavy paths, then the tree contains at least $\sim 2^k$ vertices.



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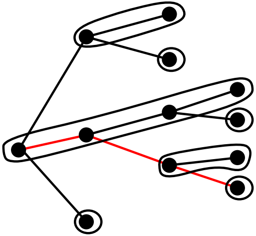
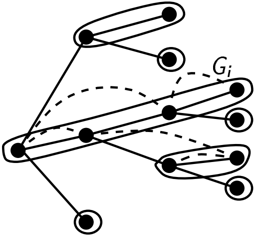


algorithm

Unbalanced trees:

- heavy-light decomposition,
- color vertices in each heavy paths (overlap graph) (first coordinate)
- use modified on-line algorithm,
- if the k-th color is used then the path from the root to the vertex intersects at least $\backsim 2^k$ heavy paths,
- the graph contains at least $\backsim 2^{2^k}$ vertices.



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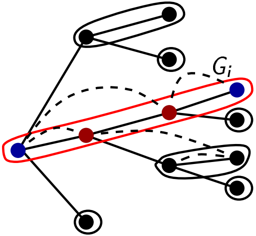

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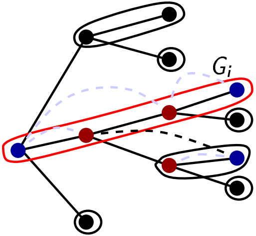

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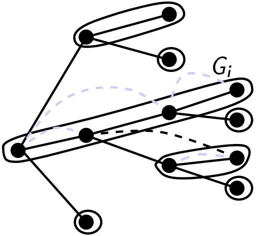

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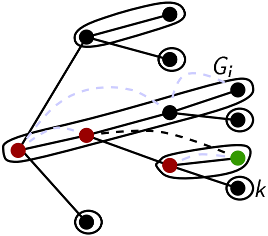

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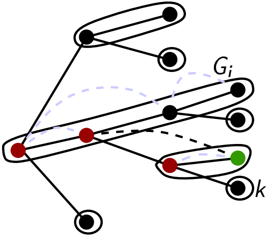


Theorem (T.K., Pawlik, Walczak, 2012)

- *There is a strategy for Algorithm that uses $O(\log k)$ colors in k rounds of the triangle-free overlap coloring game.*
- *The above strategy can be adapted to work also with heavy-light decomposition.*
- *We can color triangle-free frame intersection graph with n vertices using $O(\log \log n)$ colors.*

Question:

- Is there a strategy for Algorithm in K_ω-free overlap coloring game that uses $O(\log n)$ colors?

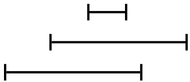
partial results

Theorem (T.K., Pawlik, Wlaczak, 2013)

There is a strategy for Algorithm that uses $O(\log n)$ *colors provided Presenter builds clean overlap graphs .*

Theorem (Kostochka, Kratochvíl, 1997)

Every K_ω *overlap graph can be colored with* $50 \cdot 2^\omega$ *colors!*



Theorem (Kostochka, Milans, 2010)

Every K_ω *clean overlap graph can be colored with* $2\omega - 1$ *colors!*

Questions:

- Does every K_ω-free segment graph (L-shaped intersection graph) partitions into $f(\omega)$ overlap game graphs?
- Does every segment graph with high chromatic number contains a large (of linear size) overlap game graph?
- (Fox, Pach) Does every K_ω-free segment/string graph contains an independent set of linear size (cn for some $c > 0$)?
- Does every K_ω-free frame intersection graph contains an independent set of linear size (cn for some $c > 0$)?